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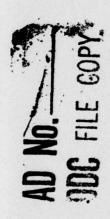
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THE GREEDY HEURISTIC APPLIED TO A CLASS OF

SET PARTITIONING AND SUBSET SELECTION PROBLEMS

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THE GREEDY HEURISTIC APPLIED TO A CLASS OF
SET PARTITIONING AND SUBSET SELECTION PROBLEMS*

by

Richard Engelbrecht-Wiggans**

Abstract

The greedy heuristic may be used to obtain approximate solutions
to integer programming problems. For some classes of problems, notably
knapsack problems related to the coin changing problem, the greedy
heuristic results in optimal solutions. However, the greedy heuristic
does quite poorly at maximizing submodular set functions.

This paper considers a class of set partitioning and subset selec-
tion problems. Results similar to those for maximizing submodular set
functions are obtained for less restricted objective functions. The ex-
ample used to show how poorly the heuristic does is motivated by a prob-
lem arising from an actual auction; the negative results are not mere
mathematical pathologies but genuine shortcomings of the greedy heuristic.

The greedy heuristic is quite successful at solving a class of knapsack problems related to the coin changing problem. Chang and Korsh [2], Hu and Lenard [5], Johnson and Kernighan [7], and Magazine, Nemhauser, and Trotter [8] show that the greedy heuristic results in optimal solutions for such problems. Problems of optimal subset selection have been studied by Boyce, Farhi, and Weischedel [1], indicating the need for a simply heuristic for obtaining approximate solutions. Fisher, Nemhauser, and Wolsey [4, 9, 10] have shown that the greedy heuristic may result in a solution for problems of maximizing submodular set functions with a value which is a relatively small fraction of the optimum.

This paper derives similar results for a wider class of set partitioning and subset selection problems. The problem is formulated in the first section of the paper. Although the motivating problem results in a set partitioning problem, the results of the later sections apply as well to a wider class of subset selection problems. The more general problem statement is given as problem II; however, most of the discussion uses examples form the context of the more restrictive problem I.

The second section considers various possible restrictions to be placed on the objective function. The conditions may be stated in terms of either of the problem statements; the two forms of the conditions are shown to be essentially equivalent. Included among the possibilities are submodular set functions and several alternatives which are relaxations of submodularity. The relative generality of the various possibilities is illustrated by a couple of simple examples.

The next two sections contain the main results of the paper. Objective functions which are "normal," "monotonic," and "discounted"

are considered first. For such cases, the greedy heuristic solution is shown to have a value of at least 1/m of the optimal value, where m is the cardinality of the largest feasible subsets. The third section concludes by presenting a class of examples for which the greedy solution value is arbitrarily little more than the bound established above.

Similar bounds may be obtained if the "discounted" condition is replaced by "variably discountedness," although now the bounds must be functions of the variable discounting functions. Again, a lower bound is derived for the greedy solution value. The section concludes by presenting a class of examples for which the greedy solution value is arbitrarily little more than this bound.

The last section is an attempt to reassure the reader that the above results are not simply pathological cases. An actual real estate auction [6] is briefly described. This real world problem is used to motivate bidding functions (of two hypothetical bidders) similar to those used to establish the tightness of the bound in sections three and four. This discussion suggests that the results are not mere mathematical pathologies and that, from many a practical viewpoint, the greedy heuristic is not a satisfactory algorithm for obtaining approximately optimal solutions to set partitioning and subset selection problems.

1. Problem Formulation

The problem is motivated by the question of Pareto optimally assigning m indivisible items among n individuals. Each individual i has a value function v_i defined over all possible subsets of the m items. The object is then to find the partition $\underline{s} = (s_1, s_2, \ldots, s_n)$ of the m items which maximizes the sum $V(\underline{s}) = \sum_{i=1}^{i=n} v_i(s_i)$ over all possible partitions.

In order to formulate this problem as an integer program, let X_j^i be zero-one variables; X_j^i is one if and only if item j is assigned to individual i. Thus, s_i is the set of j for which X_j^i is one. Let $X^i = (X_1^i, X_2^i, \ldots, X_m^i)$, and let $X = (X^1, X^2, \ldots, X^n)$. Throughout the paper, the notation will be abused slightly; in particular, $v_i(X^i)$ and $v_i(s_i)$ represent the same quantity and will be used interchangeably. Likewise, $V(\underline{s})$ and $V(X)$ represent the same quantity and will be used interchangeably.

The problem of finding a Pareto optimal assignment may now be written as the following set partitioning problem.

<u>Problem I</u>. Maximize $V(X) = \sum_{i=1}^{i=n} v_i(X^i)$ subject to all X_j^i being either zero or one; and such that

$$1) \quad \sum_{i=1}^{i=n} X^i \le 1 \quad \text{for all } j.$$

The inequality constraint assures that for feasible solutions, each item is assigned to at most one individual; for infeasible solutions, the s_i need not be disjoint. Notice that if all items have positive "marginal" value to all individuals, then each item will be assigned exactly once in any optimal solution.

Since the results of this paper hold for a wider class of problems, a more general form of the problem is stated below. The theorems are proved for the more general problem; however, all the examples used also satisfy the more restrictive formulation.

<u>Problem II</u>. Maximize $V(X)$ subject to all X_j^i are zero or one, and any constraints such that

1) any X with exactly one $X_j^i = 1$ is feasible, and

2) feasible X have at most m components $X_j^i = 1$.

It is clear that problem II is a special case of problem I. The second
formulation is one of optimal subset selection subject to an upper bound
on the cardinality of feasible subsets.

2. Restrictions on the Objective Function

Several restrictions will be considered for the objective function.
The first is simply a normalization assumption; the objective function
has a value of zero for the zero solution. The second assumption requires
that the objective is a monotonically non-decreasing function of the vector
X . An intuitive interpretation of this is that (in terms of the motivat-
ing problem) each item has a non-negative marginal value to each individual
regardless of what items the individual already has.

Several forms are considered for the third restriction. The first
is the submodularity used by Fisher, Nemhauser, and Wolsey. Next is the
subadditivity condition often used in game theory. It will not be con-
sidered explicitly in the remaining analysis, but several of the examples
have objective functions which are subadditive in addition to being dis-
counted or variably discounted.

The third form is a very special case of subadditivity; the value
of a set need only be subadditive with respect to the sum of the values
for individual items. An obvious extension of the discounted condition
is the fourth and last form, that of variable discountedness.

The conditions may be stated for either of the problem formulations.
First they are stated for the more general problem. However, they are
restated for the less general formulation since it is the motivating prob-
lem and variably discounted functions are slightly more general in this
context than in that of the more general problem.

Specifically, consider the following possible restrictions,

1) Normality: $V(0) = 0$;

2) Monotonicity: $V(X) \geq V(Y)$ whenever $X \geq Y$;

3) Submodularity: $V(X) + V(Y) \geq V(X \cup Y) + V(X \cap Y)$ for all

    X and Y ;

3') Subadditivity: $V(X) + V(Y) \geq V(X \cup Y)$ for all X and Y ;

3") Discounted: $V(X) \leq \Sigma_{i,j:X_j^i=1} V(e_j^i)$ for all X (where

    e_j^i is the unit vector with component i,j equal to one

    and all other components equal to zero); and

3*) Variably Discounted (with non-negative discount function

    D): $V(X) \leq D(|\{(i,j) : X_j^i = 1\}|) \Sigma_{i,j:X_j^i=1} V(e_j^i)$ for all X .

Alternatively, if $V(X) = \sum_{i=1}^{i=n} v_i(X^i)$, then the conditions may be stated
in terms of the individual v_i .

<u>1</u>) Normality: $v_i(\emptyset) = 0$ for all i ;

<u>2</u>) Monotonicity: $v_i(s_i) \geq v_i(t_i)$ whenever $s_i \geq t_i$ and

    for all i ;

<u>3</u>) Submodularity: $v_i(s_i) + v_i(t_i) \geq v_i(s_i \cup t_i) + v_i(s_i \cap t_i)$

    for all s_i , t_i , and for all i ;

<u>3</u>') Subadditivity: $v_i(s_i) + v_i(t_i) \geq v_i(s_i \cup t_i)$ for all s_i ,

    t_i , and for all i ;

<u>3</u>") Discounted: $v_i(s_i) \leq \Sigma_{j:j \in s_i} v_i(j)$ for all s_i and for

    all i ;

<u>3</u>*) Variably Discounted (with non-negative discount functions

    D_i): $v_i(s_i) \leq D_i(|s_i|) \Sigma_{j:j \in s_i} v_i(j)$.

There is a close connection between the two forms of the conditions.

<u>Lemma 1</u>. If $V(X) = \Sigma_{i=1}^{i=n} v_i(X^i)$, then

1) Any condition $\underline{i}$ implies the corresponding condition i ;

2) Conditions $\underline{1}$ and 2 together imply condition $\underline{2}$;

3) Condition $\underline{1}$ and any form of condition 3 together imply the corresponding form of condition $\underline{3}$. (In the case of $\underline{3}*$, the D_i may all be set equal to the D of 3*.)

<u>Proof</u>. The first claim is obvious. The remaining may be verified by considering vectors X with all but one of the subvectors X^i identically equal to zero.

The different forms of the third condition specify how nonadditive the individuals value functions may be. Fisher, Nemhauser, and Wolsey [4, 9, 10] study the performance of the greedy heuristic when the objective function satisfies conditions 1, 2, and 3. This paper obtains similar results when condition 3 is relaxed to one of the forms of 3" or 3*.

The various forms of the third condition are listed in order of increasing generality, as will be verified below. The "discounted" restriction is the special case of "variably discounted" where the discount function(s) is(are) identically equal to one. It is easy to show that normality and monotonicity together with submodularity imply subadditivity. Likewise, normality and monotonicity together with subadditivity imply discountedness.

That the reverse implications are false is verified by the following examples.

Example 1. Let $m = 3$, let $v_1(s_1)$ be the following function of $|s_1|$

$$|s_1| = 0 \quad 1 \quad 2 \quad 3$$

$$v_1(s_1) = 0 \quad 2 \quad 3 \quad 5 \, ,$$

and v_i is identically zero for all $i > 1$.

The $V(X)$ $(= v_1)$ is normal, monotonic, and subadditive. However, if $s_1 =$ items 1 and 2, and $t_1 =$ items 2 and 3, then $v_1(s_1) + v_1(t_1) = 3 + 3 = 6$, which is less than $v_1(s_1 \cup t_1) + v_1(s_1 \cap t_1) = 5 + 2 = 7$; thus V is not submodular.

Example 2. Let $m = 4$, let $v_1(s_1)$ be the following function of $|s_1|$

$$|s_1| = 0 \quad 1 \quad 2 \quad 3 \quad 4$$

$$v_1(s_1) = 0 \quad 1 \quad 1 \quad 1 \quad 4 \, ,$$

and v_i is identically zero for all $i > 1$.

This $V(X)$ $(= v_1)$ is again normal and monotonic; it is also discounted. However, since $v_1(\{1, 2, 3, 4\})$ is greater than twice $v_1(\{i, j\})$ for any distinct i and j, this value function cannot be subadditive. The following lemma has now been verified.

Lemma 2. For normal and monotonic functions, the following implications exist between the various forms of the third condition. $3 \Longrightarrow 3' \Longrightarrow 3''$ $\Longrightarrow 3*$ and $\underline{3} \Longrightarrow \underline{3}' \Longrightarrow \underline{3}'' \Longrightarrow \underline{3}*$.

3. Greedy Solutions for Discounted Functions

The main results of this paper are "tight" bounds on the ratio of the value of the greedy solution to that of the optimal solution. Each theorem is in two parts. First a lower bound on the ratio is obtained by considering the value of the objective after the first item has been assigned. Then, an appropriate example shows how to generate ratios arbitrarily close to this bound, thus establishing the tightness of the bound.

The greedy algorithm starts with all variables equal to zero and then changes the one variable which results in the largest increase in the objective function while maintaining feasibility of the solution. The greedy solution is any solution obtained by changing variables one by one until no further improvement of the objective function is possible. Notice that for monotonic objective functions, variables may be changed from zero to one, but the greedy heuristic will never change a variable from one back to zero. Thus, for problem II, the greedy solution is obtained in at most m iterations. However, the value of this solution may be only a small fraction of the optimal value as shown in the following theorem.

Theorem 1. When applied to problem II with a normal, monotonic and discounted objective function, the greedy heuristic will result in at least 1/m of the value of an optimal solution.

Proof. The monotonicity of the objective function implies that the greedy solution value is at least the value of the first item assigned. Thus it suffices to show that the most valuable single item has a value of at least of an optimal solution value. This may be verified by noting that

for an optimal solution X^* and corresponding value $V(X^*)$ the condition that the objective is discounted implies that

$$V(X^*) \leq \sum_{i,j:X^*{}_j^i=1} V(e_j^i)$$

$$\leq |\{(i,j) : X^*{}_j^i = 1\}| \text{ maximum}_{i,j} V(e_j^i)$$

$$\leq m \text{ maximum}_{i,j} V(e_j^i) .$$

Thus, $\text{maximum}_{i,j} V(e_j^i) \geq V(X^*)/m$ as desired.

The above proof verifies that the most valuable single item must be worth at least $1/m$ of the optimal solution value. The following example illustrates cases in which the most valuable item has value arbitrarily close to this bound and where the remaining items add arbitrarily little to the greedy solution value.

Example 3. Let $n \geq 2$, $m \geq 2$, $0 < d < e/m$, and

$$v_1(1) = 1+d , \quad v_1(s_1) = d - 1 + |s_1| \text{ if } |s_1| > 1 \text{ and } 1 \in s_1 ,$$

$$\text{and } v_1(s_1) = |s_1| \text{ if } 1 \notin s_1 ;$$

$$v_2(s_2) = d|s_2| \text{ if } 1 \notin s_2 , \text{ and } v_2(s_2) = 1 - d + d|s_2| \text{ if } 1 \in s_2 ;$$

and $\quad v_i(s_i) = 0 \text{ for all } s_i \text{ for all } i > 2 .$

Note that these v_i are not only normal, monotonic, and discounted, but that they are also subadditive.

The optimal assignment for example 3 is $s_1 = \{2, 3, \ldots, m\}$ and $s_2 = \{1\}$, with an associated optimal value of m. The greedy heuristic,

on the other hand, will first assign item 1 to the first individual since maximum$_{i,j}V(e_j^1) = 1+d = v_1(e_1^1)$. Once $X_1^1 = 1$, the marginal value to the first individual of any second item is zero, whereas the marginal value to the second individual for additional items is d . Thus all of the remaining items will be assigned to the second individual. This results in $s_1 = \{1\}$ and $s_2 = \{2, 3, \ldots, m\}$. The resulting greedy solution value is $1 + md$, which by the definition of d is less than $1+e$.

Since the optimal value is m , the greedy solution has a value of less than e in excess of $1/m$ of the optimal value. This proves the following theorem.

Theorem 2. When problem II has a normal, monotonic, and discounted objective function, then for any $e > 0$, there is an example (based on example 3) such that the following relationship exists between the optimal value V^* and the greedy solution value $V_g : V^*/m \leq V_g < V^*/m + e$.

4. Greedy Solutions for Variably Discounted Functions

Results similar to the above may be obtained for the case of variably discounted functions. However, in this case, the bound must be in terms of the discounting functions. Since condition 3* (with a single discount function) is the special case of condition 3* with all discount functions D_i equal, the following results will be in terms of the latter and more context.

Although the discount functions D_i may be any functions such that condition 3* is satisfied, it will be assumed that the discount functions actually used are the minimum possible such functions. It re-

mains to be verified that there exist minimum discount functions; the verification follows.

For any fixed $k \leq m$, there is a finite number of subsets containing exactly k of the m items. Now let $D_i(k)$ be the maximum$_{s_i : |s_i| = k} v_i(s_i) / \Sigma_{j : j \in s_i} v_i(j)$. This maximum must exist, and the resulting D_i is the desired discount function for individual i. Thus a minimum discount function exists for each individual.

For actual data, the minimum discount function may be extremely difficult to calculate; indeed the work involved may be comparable to solving the set partitioning problem exactly. In this case, some approximately minimum discount function, perhaps determined using any structure that values might have, must be used. The following results are in terms of the minimum discount function; equally correct (but not "tight") bounds result from using non-minimal discount functions.

Theorem 3. When problem II has a normal, monotonic, and variably discounted objective function (with discount functions D_i), then the greedy heuristic will result in a value V_g satisfying the following.
$V_g \geq V^* / \text{maximum}_{\underline{k}} \Sigma_{i=1}^{i=n} k_i D_i(k_i)$ where $\underline{k} = (k_1, k_2, \ldots, k_n)$ is any vector with non-negative integers as coordinates such that $\Sigma_{i=1}^{i=n} k_i = m$. (The vector $\underline{k}$ may be viewed as specifying the number of items assigned to each individual.) And V^* denotes the optimal value.

Proof. As with the proof to theorem 1, it is only necessary to verify that the most valuable single item has at least the above specified value. This is true, since for an optimal solution X^*,

$$V(X^*) \leq \Sigma_{i=1}^{i=n} D_i(|\{j : X^{*i}_j = 1\}|) \Sigma_{j:X^{*i}_j=1} V(e^i_j)$$

$$\leq \text{maximum}_{i,j} V(e^i_j) \Sigma_{i=1}^{i=n} |\{j : X^{*i}_j = 1\}| D_i(|\{j : X^{*i}_j = 1\}|)$$

$$\leq \text{maximum}_{i,j} V(e^i_j) \text{maximum}_{\underline{k}} \Sigma_{i=1}^{i=n} k_i D_i(k_i) \ .$$

Solving the inequality for " $\text{maximum}_{i,j} V(e^i_j)$ " completes the proof.

The following example shows that for discount functions not uniformly bounded by one, the greedy heuristic may do arbitrarily poorly.

Example 4. Consider example 3 except $v_1(s_1)$ is now equal to the variable $v^* > m$ when $|s_1| = m$.

As soon as v^* exceeds m , there is a $d > 0$ such that $v^* > m+d$. Thus, $D_1(m) = v^*/(m+d)$ is greater than one and v_1 is no longer a discounted function. (Note that all other $D_i(k) \leq 1$.) The new optimal solution is $s_1 = \{1, 2, \ldots, m\}$ and has a value of v^* ; the greedy solution remains unchanged. As v^* goes to infinity, the greedy solution is arbitrarily bad when compared to the optimal.

Since all $D_i(k) \leq 1$ except $D_1(m) = v^*/(m+d) \geq 1$, the sum $\Sigma_{i=1}^{i=n} k_i D_i(k_i)$ is maximum when $k_1 = m$ and all other $k_i = 0$. The corresponding value of the sum is $mv^*/(m+d)$. Using this maximum value, the above theorem assures that the greedy solution value is at least $v^*/(mv^*/(m+d)) = 1 + d/m$. However, for $d < e/m$, $1+md$ is less than $1+e$ and therefore less than $1+d/m+e$. Thus, for any $e > 0$, this example results in a greedy solution value of less than e in excess of the lower bound established in theorem 3. This, together with theorem 2, proves the following theorem.

__Theorem 4__. When problem II has a normal, monotonic, and variably dis-
counted (with minimum discount functions D_i) objective function, then
for any $e > 0$, there is an example such that the resulting greedy so-
lution value is less than the maximum of $e + V^*/m$ and
$e + V^*/\text{maximum}_{\underline{k}} \Sigma_{i=1}^{i=n} k_i D_i(k_i)$, where V^* is the value of an optimal so-
lution and $\underline{k}$ is any vector of non-negative integers such that $\Sigma_{i=1}^{i=n} k_i = m$.

Thus the lower bound on the performance of the greedy heuristic is a "tight"
bound.

5. Relation of Examples to an Actual Problem

Although examples 3 and 4 are constructed to show how poorly the
greedy algorithm may do, the examples are motivated by a set partition-
ing problem arising from an actual auction [6]. In this particular auc-
tion, a bank is selling four plots of land; three contiguous and roughly
similar plots, and one larger separate plot (which borders on one of the
city's school properties). The bank accepted bids on single plots, on
the three contiguous plots as a set, and on all four plots as a set.
This is not very different from allowing bids on all possible subsets of
the four plots.

Consider two hypothetical potential bidders. The first is a de-
veloper wishing to build one apartment house. The sizes of the bids on
single plots reflect the sizes of the plots and the size of the largest
apartment house which may be built. A larger building may be built on
several contiguous plots, and thus the value function may be additive
for all subsets of the three smaller properties (assuming the three plots
are pairwise adjacent). However, two non-contiguous plots are valued

very little more than the most valuable single plot in the set.

The second hypothetical bidder, perhaps the city government, is really only interested in the larger plot and submits very small marginal bids on the remaining plots. Thus, the resulting bids might be like the value function in example 3.

Unfortunately, if the bank uses the greedy algorithm to decide how the plots are sold, the large plot is sold to the first individual and the remaining plots to the second individual. The resulting revenue is only about one fourth that obtained from selling the larger plot to the second individual and the remaining plots to the first individual.

This example seems plausible enough that it cannot be dismissed as a mere mathematical pathology. It must be concluded that the greedy heuristic may be very inefficient at obtaining an optimal solution value. Notice however, that the results are very sensitive to the data of the examples. A change of d in a few of the values might result in a greedy solution which is an optimal solution. One possibility for further research is to determine how and by how much any given problem should be perturbed in the search for a near optimal greedy solution.

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